# Homogeneous symplectic manifolds of Poisson-Lie groups

**P. Baguis**[1,2]

Université Libre de Bruxelles
Campus Plaine, CP 218 Bd du Triomphe
1050, Brussels, Belgium

**Abstract.** Symplectic manifolds which are homogeneous spaces of Poisson-Lie groups are studied in this paper. We show that these spaces are, under certain assumptions, covering spaces of dressing orbits of the Poisson-Lie groups which act on them. The effect of the Poisson induction procedure on such spaces is also examined, thus leading to an interesting generalization of the notion of homogeneous space. Some examples of homogeneous spaces of Poisson-Lie groups are discussed in the light of the previous results.

*Key-words*: Poisson-Lie groups, induction of Poisson actions, momentum mapping, homogeneous spaces

*MSC 2000*: 53C15, 53D17, 53D20

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[1] e-mail: pbaguis@ulb.ac.be

[2] Research supported by the "Communauté française de Belgique", through an "Action de Recherche Concertée de la Direction de la Recherche Scientifique".

## 1. Introduction

Homogeneous symplectic manifolds are, under reasonable conditions, locally isomorphic to coadjoint orbits and their relation to the theory of unitary irreducible representations of Lie groups has been very early recognized [**12**]. This kind of symplectic manifolds (together with the coadjoint orbits of Lie groups, which are a special case), is perhaps the most important non-trivial class of geometrically quantizable symplectic manifolds in the Kirillov-Kostant-Souriau program [**11**], [**12**], [**16**]. A fundamental ingredient of this approach is the existence of an equivariant momentum map for the symplectic action on the homogeneous symplectic manifold. Then, it turns out that such a symplectic manifold is a covering space of a coadjoint orbit of the group. This is essentially a "Hamiltonian" classification of the homogeneous symplectic manifolds.

Recently, an analogous study but in a different context, has been initiated by [**6**], [**9**], [**14**] for Poisson-Lie groups acting on Poisson manifolds. In particular, a correspondence between Poisson homogeneous $G$-spaces, where $G$ is a Poisson-Lie group, and Lagrangian subalgebras of the double $D(\mathfrak{g})$ of the tangent Lie bialgebra of $G$, has been established in [**6**].

In the present article, we will turn our attention again to the Hamiltonian point of view but for Poisson-Lie groups this time. We first establish the exact Poisson-Lie analog of homogeneous symplectic manifolds: a symplectic manifold on which a Poisson-Lie group acts transitively through a Poisson action admitting an equivariant momentum map, in the sense of [**13**], is a covering space of a dressing orbit of the Poisson-Lie group (see Proposition 2.2 below). Notice here that it makes no sense to replace the symplectic manifold by a Poisson one, because a transitive action of a Poisson-Lie group on a Poisson manifold has never a momentum map unless the Poisson manifold contains only one symplectic leaf, therefore is symplectic.

The case of non-equivariant momentum maps needs special attention in the Poisson-Lie case since, unlike the symplectic case, the lack of equivariance now is not automatically adjusted. We address this issue in section 3.

We also examine the effects of the Poisson induction procedure, introduced in [**2**], on a symplectic manifold which is a homogeneous Hamiltonian space of a

Poisson-Lie group. The result depends on the circumstance and the procedure leads either to a homogeneous space or to an almost homogeneous space. The later is introduced in Definition 2.3; actually, this notion emerges naturally in the induction procedure and its meaning is that the almost homogeneous space is generated by a discrete (eventually finite) subset through the action. If this subset reduces to a point, then we obtain a homogeneous space.

Some examples are finally discussed in section 5. More precisely, we describe situations in which one can have a transitive Poisson action on a symplectic manifold admiting a momentum map in the sense of [**13**] and we give partial solutions to the difficult problem of equivariance. This progressively leads to a Hamiltonian description of the cells of a Bruhat decomposition for coadjoint orbits of a certain type. We finally endow a semi-direct product $G = K \times_\rho V$ with a Poisson-Lie structure that has the following property: the corresponding dressing orbits of $G$ in $G^*$ can be obtained by Poisson induction on coadjoint orbits of certain subgroups of $K$.

**Conventions.** If $(P, \pi_P)$ is a Poisson manifold, then $\pi_P^\sharp \colon T^*P \to TP$ is the map defined by $\alpha(\pi_P^\sharp(\beta)) = \pi_P(\alpha, \beta), \forall \alpha, \beta \in T^*P$. Let now $\sigma \colon G \times P \to P$ (resp. $\sigma \colon P \times G \to P$) be a left (resp. right) Poisson action of the Poisson-Lie group $(G, \pi_G)$ on $(P, \pi_P)$, and let us denote by $\sigma(X)$ the infinitesimal generator of the action and by $G^*$ the dual group of $G$. Then, we say that $\sigma$ is Hamiltonian if there exists a differentiable map $J \colon P \to G^*$, called momentum mapping, satisfying the following equation, for each $X \in \mathfrak{g}$:

$$\sigma(X) = \pi_P^\sharp(J^* X^l) \quad (resp. \quad \sigma(X) = -\pi_P^\sharp(J^* X^r)).$$

In the previous equation $X^l$ (resp. $X^r$) is the left (resp. right) invariant 1-form on $G^*$ whose value at the identity is equal to $X \in \mathfrak{g} \cong (\mathfrak{g}^*)^*$. The momentum mapping is said to be equivariant, if it is a morphism of Poisson manifolds with respect to the Poisson structure $\pi_P$ on $P$ and the canonical Poisson structure on the dual group of the Poisson Lie group $(G, \pi_G)$. Left and right infinitesimal dressing actions $\lambda \colon \mathfrak{g}^* \to \mathfrak{X}(G)$ and $\rho \colon \mathfrak{g}^* \to \mathfrak{X}(G)$ of $\mathfrak{g}$ on $G^*$ are defined by

$$\lambda(\xi) = \pi_G^\sharp(\xi^l) \quad \text{and} \quad \rho(\xi) = -\pi_G^\sharp(\xi^r), \quad \forall \xi \in \mathfrak{g}^*.$$

Similarly, one defines infinitesimal left and right dressing actions of $\mathfrak{g}$ on $G^*$. In the case where the vector fields $\lambda(\xi)$ (or, equivalently, $\rho(\xi)$) are complete for all $\xi \in \mathfrak{g}^*$, we have left and right actions of $(G^*, \pi_{G^*})$ on $(G, \pi_G)$ denoted also by $\lambda$ and $\rho$ respectively, and we say that $(G, \pi_G)$ is a complete Poisson-Lie group.

## 2. The equivariant Poisson-Lie case

In the symplectic context, the following is well known. Let $(M, \omega)$ be a symplectic manifold and $\sigma: G \times M \to M$ a symplectic action admitting the momentum mapping $J: M \to \mathfrak{g}^*$, that is the infinitesimal generator of the action corresponding to the element $X \in \mathfrak{g}$ is equal to the Hamiltonian vector field corresponding to the function $J^*X \in C^\infty(M)$, where we regard the pull-back $J^*$ as a linear map $\mathfrak{g} \to C^\infty(M)$. Assume that the momentum map is equivariant, that is $J \circ \sigma_g = \mathrm{Coad}(g) \circ J, \forall g \in G$ and that the action $\sigma$ is transitive. Then:

**2.1 Theorem ([12]).** *Under the assumptions above, there exists an element* $\mu_0 \in \mathfrak{g}^*$ *such that* $M$ *be a covering space of the coadjoint orbit* $\mathcal{O}_{\mu_0} = G \cdot \mu_0$ *and* $J: M \to \mathcal{O}_{\mu_0}$ *be a morphism of Hamiltonian G-spaces.*

When the momentum map $J$ is not equivariant, one can consider the bilinear map $\gamma: \mathfrak{g} \times \mathfrak{g} \to C^\infty(M)$ given by $\gamma(X, Y) = \{J^*X, J^*Y\} - J^*[X, Y]$, which is a constant function on $M$ for each $X, Y \in \mathfrak{g}$ and defines a 2-cocycle $\mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$. This completely determines a central extension $\tilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ and a symplectic action on $M$ of the connected and simply connected Lie group $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{g}}$. This action admits now an equivariant momentum map and the Theorem 2.1 can be applied.

We consider now a symplectic manifold $P$, where the symplectic structure is described by a non-degenerate Poisson tensor $\pi_P$, a Poisson-Lie group $(G, \pi_G)$ and a left Poisson action $\sigma: G \times P \to P$. We make the assumption that the action $\sigma$ admits an equivariant momentum map $J: P \to G^*$. In that case, one says that $(P, \pi_P)$ is a *Hamiltonian Poisson-Lie G-space.* One has:

**2.2 Proposition.** *If the action* $\sigma$ *is transitive, then there exists an element* $u_0 \in G^*$ *such that the symplectic manifold* $P$ *be a covering space of the left dressing*

*orbit $\mathcal{O}^l_{u_0}$ of $u_0$ and the map $J: P \to \mathcal{O}^l_{u_0}$ be a morphism of Hamiltonian Poisson-Lie $G$-spaces.*

Proof. Let $p_0 \in P$ and $u_0 = J(p_0) \in G^*$. Then, by the equivariance (and infinitesimal equivariance) of the momentum mapping, we find that $J: P \to \mathcal{O}^l_{u_0}$ is a surjective submersion, where $\mathcal{O}^l_{u_0}$ is the left dressing orbit of $u_0$.

Consider now the equation $T_pJ(v) = 0$ for $v \in T_pP$. One can write $v = \sigma(X)_p, X \in \mathfrak{g}$ because $P$ is homogeneous space of $G$. Then $\lambda(X)(J(p)) = 0$ and consequently $(J^*X^l)(\sigma(Y)_p) = 0$ for each $Y \in \mathfrak{g}$. Using again the fact that $P$ is homogeneous and that $J: P \to \mathcal{O}^l_{u_0}$ is a surjective submersion, we have the result. ■

If we write $P = G/G_{p_0}$ and $\mathcal{O}^l_{u_0} = G/G_{u_0}$, where $G_{p_0}$ and $G_{u_0}$ are the isotropy subgroups of $p_0$ and $u_0$ for the corresponding actions, then the map $J: P \to \mathcal{O}^l_{u_0}$ can be writen as $J([g]_{p_0}) = [g]_{u_0}$, where $[g]_{p_0}$ and $[g]_{u_0}$ denote the equivalence classes of $g \in G$ under the equivalence relations defined by the subgroups $G_{p_0}$ and $G_{u_0}$. Furthermore, one has $G_{p_0} \subset G_{u_0}$ and the fibre of $J$ is exactly $G_{u_0}/G_{p_0}$.

There exists a generalization of the notion of homogeneous space which arises naturally in the induction procedure, as we will see later. We give the following definition:

**2.3 Definition.** *Let $P$ be a differentiable manifold on which the Lie group $G$ acts smoothly. We will say that $P$ is an almost homogeneous space of $G$ if there exists a discrete subset $\Sigma \subset P$ such that $G \cdot \Sigma = P$.*

Otherwise stated, $P$ is almost homogeneous, if the set of $G$-orbits in $P$ is discrete. In particular, when $\Sigma$ is a one-point set, $P$ is homogeneous.

Assume now that $(P, \pi_P)$ is a symplectic manifold which is an almost homogeneous space of the Poisson-Lie group $(G, \pi_G)$ for a left Hamiltonian Poisson action of $G$ on $P$. Then, it is immediate from Proposition 2.2 that all the open orbits of $G$ in $P$ are covering spaces of left dressing orbits of $G$ in $G^*$. In particular, if the topology on $\Sigma$, viewed as a quotient space $P/G$, is the discrete one, then the manifold $P$ if "foliated" by a discrete set of open submanifolds, each of them is a covering space of a left dressing orbit in $G^*$.

As an example, let us discuss the following situation coming from the com-

pletely symplectic setting. We take $P = \mathbf{R}^2$ equipped with its canonical symplectic structure and $G$ equal to the semidirect product between $K = \mathbf{R}^\star_+$ (non-zero positive real numbers) and $V = \mathbf{R}$ through the representation $K \to GL(V)$ given by $r \to \frac{1}{r}$ [**3**]. Then, $(r,a)\cdot(x,y) = (rx, \frac{1}{r}y + a)$ is a symplectic action which admits the equivariant momentum map $J\colon P \to \mathfrak{g}^* \cong \mathbf{R}^2$ given by $J(x,y) = (-xy, x)$. Using the induction techniques of [**3**] for coadjoint orbits of semidirect products, one easily finds that the only coadjoint orbits of $G$ are either a point or the cotangent bundle $T^*V \cong \mathbf{R}^2$ with its canonical symplectic structure. On the other hand, the space $P$ is almost homogeneous since it is generated by the set $\Sigma = \{(-1,0),(0,0),(1,0)\}$ through the action of $G$. Furthermore, there exist two open orbits in $P$, the open half-planes $P_\pm = \mathbf{R}^\star_\pm \times \mathbf{R}$. According to the previous discussion on almost homogeneous spaces, $P_\pm$ coincide with coadjoint orbits of $G$. Apparently, nothing can be said about the $G$-orbit of $(0,0)$ which coincides with the $y$-axis and therfore is 1-dimensional.

## 3. The non-equivariant Poisson-Lie case

We are now placed in the case where the Poisson action $\sigma\colon G \times P \to P$ admits a non-equivariant momentum map $J\colon P \to G^*$. Without loss of generality, we can assume that there exists a point $x_0 \in P$ such that $J(x_0) = e^*$, the identity of $G^*$; let $\gamma = J_*\pi_P(x_0) \in \Lambda^2\mathfrak{g}^*$. For each $X, Y \in \mathfrak{g}$, we consider the function $\mu(X,Y) \in C^\infty(P)$ defined by

$$\mu(X,Y) = \pi_P(J^*X^l, J^*Y^l) - J^*(\pi_{G^*}(X^l, Y^l)). \tag{3.1}$$

The function $\mu(X,Y)$ controls the equivariance of $J$ because $\mu(X,Y) = 0$ forall $X, Y \in \mathfrak{g}$ if and only if $J$ is equivariant.

Let now $\pi_J = \pi_{G^*} + \gamma^r$. Then, it can be proved [**13**] that $J\colon (P,\pi_P) \to (G^*, \pi_J)$ is a Poisson map. But this means that

$$\mu(X,Y)(p) = (\mathrm{Ad}(J(p)^{-1})\gamma)(X,Y), \forall X, Y \in \mathfrak{g}, p \in P.$$

We see that if $\gamma$ is Ad-invariant, then $\mu(X,Y)$ is a constant function on $P$. But there is more than this:

**3.1 Proposition.** *If $\gamma$ is Ad-invariant, then $\gamma$ is a real-valued 2-cocycle on the Lie algebra $\mathfrak{g}$.*

We refer the reader to [4] for the proof of this proposition. Assuming now that $\gamma$ is Ad-invariant, let us consider the central extension of the Lie bialgebra $\mathfrak{g}$ defined by the cocycle $\gamma$ and the zero derivation on $\mathfrak{g}^*$. For the reader's convenience, we recall from [5] that a central extension of a Lie bialgebra $\mathfrak{g}$ is an exact sequence $0 \longrightarrow \mathbf{R} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{j} \mathfrak{g} \longrightarrow 0$, such that $i$ and $j$ be morphisms of Lie bialgebras and $i(\mathbf{R})$ be contained in the center of $\hat{\mathfrak{g}}$. If $Z^2(\mathfrak{g}, \mathbf{R})$ and $\mathcal{D}er(\mathfrak{g}^*)$ stand for the space of real-valued 2-cocycles over $\mathfrak{g}$ and the space of derivations on the Lie algebra $\mathfrak{g}^*$, then such an extension is completely determined by a pair of elements $(\gamma, f) \in Z^2(\mathfrak{g}, \mathbf{R}) \times \mathcal{D}er(\mathfrak{g}^*)$ which are Drinfeld-compatible:

$$f^*([X,Y]) - [f^*(X), Y] - [X, f^*(Y)] = \mathrm{coad}(\gamma^\sharp(X))(Y) - \mathrm{coad}(\gamma^\sharp(Y))(X). \quad (3.2)$$

Returning to our case, the pair $(\gamma = J_* \pi_P(x_0), f = 0)$ defines actually a central extension of $\mathfrak{g}$ because the Ad-invariance of $\gamma$ gives the Drinfeld-compatibility with $f = 0$. If $\hat{\mathfrak{g}}$ is the extended Lie bialgebra, we have an infinitesimal left action $\hat{\sigma}: \hat{\mathfrak{g}} \to \mathcal{X}(P)$ given by $\hat{\sigma}(X, a) = \sigma(X)$. This action is Poisson giving thus rise to a Poisson action, still denoted by $\hat{\sigma}$, of the connected and simply connected Poisson-Lie group $\hat{G}$ having $\hat{\mathfrak{g}}$ as tangent Lie bialgebra, on the Poisson manifold $(P, \pi_P)$. Under the isomorphism $\hat{G} \cong \tilde{G} \times \mathbf{R}$, where $\tilde{G}$ is the universal covering space of $G$, one has:

**3.2 Theorem.** *The differentiable map $\hat{J}: P \to \hat{G}^*$ given by*

$$\hat{J}(p) = (J(p), 1), \forall p \in P \quad (3.3)$$

*is an equivariant momentum map for the Poisson action $\hat{\sigma}$.*

<u>*Proof.*</u> We first note that the Poisson structure on $\hat{G}^* \cong G^* \times \mathbf{R}$ is given by

$$\pi_{\hat{G}^*}(v, a) = \pi_{G^*}(v) + a\gamma^r(v), \forall (v, a) \in \hat{G}^*.$$

The rest of the proof consists of a calculation of the function $\mu(X, Y)$ for the momentum map $\hat{J}$ and it is ommited here. ∎

## 4. Poisson induction of homogeneous symplectic manifolds

Symplectic induction is a procedure by means of which one can induce, from precise initial data, symplectic structures and Hamiltonian actions to bigger manifolds and has several interesting applications [**10**], [**7**], [**8**], [**3**]. We know [**4**] that the induction procedure has a natural Poisson analog and, in particular, that the Poisson induced of a symplectic manifold is also a symplectic manifold carrying a Hamiltonian Poisson action. Here we will study the effect of the Poisson induction procedure on a homogeneous symplectic manifold, a problem whose even the completely symplectic aspects are not yet known.

Let us recall from [**4**] the elements of the Poisson induction which are necessary for the understanding of what follows. We consider a left Poisson action $\sigma\colon (H, \pi_H) \times (P, \pi_P) \to (P, \pi_P)$ admitting the equivariant momentum mapping $J\colon P \to H^*$ and assume that $(H, \pi_H)$ is a Poisson-Lie subgroup of the Poisson-Lie group $(G, \pi_G)$. In order to simplify technically the discussion, we make the assumption that $(G, \pi_G)$ is simply connected and complete. If $(D(G), \pi_+)$ is the double group of $G$ equipped with its natural symplectic structure [**13**], let $(\tilde{P}, \pi_{\tilde{P}}) = (P, \pi_P) \times (D(G), \pi_+)$. Then the map $\tilde{\sigma}\colon H \times \tilde{P} \to \tilde{P}$ given by

$$\tilde{\sigma}(h, (p, d)) = (\sigma(\lambda_{i^* u^{-1}}(h), p), d[\lambda_{i^* u^{-1}}(h)]^{-1}), \forall h \in H, (p, d) \in \tilde{P} \tag{4.1}$$

where $i^*\colon G^* \to H^*$ is the projection of the dual groups induced by the inclusion $i\colon H \hookrightarrow G$, is a left Poisson action of $H$ on $\tilde{P}$ admitting the equivariant momentum map $\tilde{J}\colon \tilde{P} \to H^*$ with

$$\tilde{J}(p, d) = J(p)(i^* u^{-1}). \tag{4.2}$$

If $e^* \in H^*$ is the unit element, we obtain by Marsden-Weinstein reduction the induced manifold as

$$P_{ind} = \tilde{J}^{-1}(e^*)/H. \tag{4.3}$$

The group $G$ acts on $P_{ind}$ as follows: if $[(p, gu)]$ is the equivalence class of $(p, gu) \in \tilde{J}^{-1}(e^*)$ in the quotient (4.3), then

$$k \cdot [(p, gu)] = [(p, \lambda_{\rho_{g^{-1}}(u)}(k) gu)] \tag{4.4}$$

for all $k \in G$. This action is Poisson and admits an equivariant momenutm mapping.

The above construction works even when $(P, \pi_P)$ is a general Poisson manifold, but here we are interested in the case where $P$ is symplectic. Then, $P_{ind}$ is also symplectic and the following question arises naturally: what can we tell about $P_{ind}$ when $P$ is a homogeneous space of $H$?

Let us fix an element $[(p_0, g_0u_0)] \in P_{ind}$ and consider an arbitrary element $[(p, gu)] \in P_{ind}$. Then, assuming that $H$ acts transitively on $P$, we conclude that there exist elements $h \in H$ and $u^1 \in H^\circ$ such that

$$\left\{\begin{array}{lcl} p & = & (\sigma_J)_h(p_0) \\ u & = & \rho_{h^{-1}}(u_0u^1). \end{array}\right. \tag{4.5}$$

Here $H^\circ = \ker(i^*)$ and $\sigma_J$ is the action of $H$ on $P$ defined as $(\sigma_J)_h(p) = \sigma_{\lambda_{J(p)^{-1}}(h)}(p)$. We want now to solve the equation $k \cdot [(p_0, g_0u_0)] = [(p, gu)]$ with respect to $k \in G$ for given $[(p_0, g_0u_0)]$ and arbitrary $[(p, gu)]$. In view of the transitivity of the action $\sigma$ and of (4.4), this equation is equivalent to

$$\left\{\begin{array}{lcl} p & = & (\sigma_J)_h(p_0) \\ g & = & \lambda_{\rho_{g_0^{-1}}(u_0)}(k)g_0h^{-1} \\ u & = & \rho_{h^{-1}}(u_0). \end{array}\right. \tag{4.6}$$

The last equations make clear that, generally, one can nothing say about the action of $G$ on $P_{ind}$, despite the transitivity of $\sigma$. Let us make the following assumption: $u_0H^\circ \subset H \cdot u_0$, where the dot on the right hand side means left dressing transformations. If $J(p_0) = v_0$, one then observes:

- if $h_1 \in H$ is an element for which $\lambda_{h_1}(u_0) = u_0u^1$, then $h_1 \in H_{v_0}$, where $H_{v_0}$ represents the isotropy subgroup of $v_0 \in H^*$ with respect to the left dressing tranformations;
- $u_0u^1 = \rho_{\tilde{h}_1^{-1}}(u_0)$, where $\tilde{h}_1^{-1} = \lambda_{J(p_0)}(h_1)$.

Consequently, the equations (4.5) become

$$\left\{\begin{array}{lcl} p & = & (\sigma_J)_{h\tilde{h}_1}(p_0') \\ u & = & \rho_{(h\tilde{h}_1)^{-1}}(u_0), \end{array}\right.$$

where $p'_0 = \sigma_{h_1^{-1}}(p_0) \in J^{-1}(v_0)$. Combining with equations (4.6) we obtain the following theorem:

**4.1 Theorem.** *Assume that the symplectic manifold $(P, \pi_P)$ is a homogeneous Hamiltonian Poisson-Lie space of the Poisson-Lie group $(H, \pi_H)$, viewed as a Poisson-Lie subgroup of $(G, \pi_G)$. Then, the Poisson induced symplectic manifold $(P_{ind}, \pi_{ind})$ has the following structure:*

- *if $u_0 H^\circ \subset H \cdot u_0$, then $P_{ind}$ is an almost homogeneous Hamiltonian Poisson-Lie space of $G$;*
- *if $u_0 H^\circ \subset H_{p_0} \cdot u_0$, where $H_{p_0}$ is the isotropy subgroup of $p_0$ for the action $\sigma$ of $H$ on $P$, then $P_{ind}$ is a homogeneous Hamiltonian Poisson-Lie space of $G$ and, in view of Proposition 2.2, a covering space of a left dressing orbit of $G$ in $G^*$.*

## 5. Applications

**(1)** Consider two complete Poisson-Lie groups $(H, \pi_H)$ and $(G, \pi_G)$ and an injective morphism of Poisson-Lie groups $f\colon H \to G$ which is an immersion at the identity. Then, the linear map $\sigma\colon \mathfrak{g}^* \to \mathfrak{X}(H)$ given by

$$\sigma(\xi) = \pi_H^\sharp(f^* \xi^l)$$

is an infinitesimal Poisson action admitting $f$ as equivariant momentum map. In the complete case we are studying, the corresponding Poisson action $\sigma$ of $G^*$ on $H$ is given by the equation $\sigma_u(h) = \lambda_{f^* u}(h)$, where $f^*\colon G^* \to H^*$ is the morphism of the dual groups induced by $f$. In view of Proposition 2.2, one obtains:

**5.1 Corollary.** *The orbit of $h \in H$ under the left dressing action of $H^*$ is a homogeneous Hamiltonian Poisson-Lie $G^*$-space and, at the same time, a covering space of the orbit of $f(h) \in G$ under the left dressing action of $G^*$. In particular, these orbits have the same dimension.*

**(2)** If $(P, \pi_P)$ is a non-trivial homogeneous Hamiltonian Poisson-Lie space of $(G, \pi_G)$, then there is no point of $P$ whose image under the momentum map $J$ could be the identity of the group $G^*$. Indeed, if such a point existed, then $P$

should be locally isomorphic to a point according to Proposition 2.2, which is a contradiction.

Consider now a simply connected symplectic manifold $P$ and a transitive left Poisson action $\sigma: G \times P \to P$ of the Poisson-Lie group $(G, \pi_G)$. Then, for each point $x_0 \in P$ there exists a momentum mapping $J: P \to G^*$ for this action [**13**], such that $J(x_0) = e^*$, the identity of $G^*$. Then, according to the previous argument, this momentum map cannot be equivariant. Assume instead that there exists an element $u_0 \in G^*$, such that the bivectors $\pi_P$ and $\pi_{G^*}$ be $J_0$-related at the point $x_0$, where $J_0 = L_{u_0} \circ J$. Then, $J_0$ is a Poisson morphism between $(P, \pi_P)$ and $(G, \pi_{G^*})$ and hence an equivariant momentum map for the action $\sigma$. In this case, $P$ is the universal covering space of a left dressing orbit of $G$ in $G^*$.

Staying always in the case of a transitive left Poisson action on a simply connected symplectic manifold, assume that there exists a point $x_0 \in P$ such that the annihilator of the isotropy subalgebra $\mathfrak{g}_{x_0}$ (with respect to the action $\sigma$) be contained in the center of the Lie algebra $\mathfrak{g}^*$. Then, the element $\gamma \in \Lambda^2\mathfrak{g}$ of Proposition 3.1 is invariant under the adjoint action of $G^*$ but not zero, because of the transitivity of the action. Then, by Theorem 3.2 and Proposition 2.2, $P$ is the universal covering space of a left dressing orbit in the dual group of an appropriate central extension of $G$.

**(3)** We consider now the case where $(H, \pi_H)$ is a Poisson-Lie group of $(G, \pi_G)$, with $\pi_G(g) = R_g r - L_g r$ and $\pi_H = 0$, where $r \in \Lambda^2\mathfrak{g}$ has the property $\mathrm{Ad}(g)[r, r] = [r, r]$. Otherwise stated, $\pi_G$ is exact and $\mathrm{Ad}(h)r = r, \forall h \in H$. Let $\mathcal{O} = G/H$ and $\bar{\rho}: \mathcal{O} \times G^* \to \mathcal{O}$ the right action of $G^*$ on the Poisson manifold $\mathcal{O}$ [**15**], obtained projecting the right dressing transformations of $G^*$ in $G$. Then, for each point $x_0 = [g_0] \in \mathcal{O}$, the orbit $x_0 \cdot G^*$ coincides with the symplectic leaf through $x_0$. Pick an element $g_0 \in G$ such that the orbit of $x_0$ be simply connected. Then, the action $\bar{\rho}$ restricts to a right Poisson action of $G^*$ on $x_0 \cdot G^*$ for which there exists a momentum mapping $J: x_0 \cdot G^* \to G$ with $J(x_0) = e$. By construction, the tangent of $J$ at $x_0$ is given by $J_{*x_0}(\bar{\rho}(\eta)_{x_0})(\xi) = -\omega_{x_0}(\bar{\rho}(\eta)_{x_0}, \bar{\rho}(\xi)_{x_0}), \forall \eta, \xi \in \mathfrak{g}^*$, where $\omega$ is the symplectic structure of the symplectic leaf $x_0 \cdot G^*$. Assume now that the Lie algebra $\mathfrak{h}$ of $H$ has center of dimension 0 or 1. Then, one finds

$$J_{*x_0}(\bar{\rho}(\eta)_{x_0}) = R_{g_0^{-1}}\rho(\eta)_{g_0}$$

which has as direct consequence that $J_{*x_0}\pi(x_0) = R_{g_0^{-1}}\pi_G(g_0)$, where $\pi$ is the Poisson structure of $\mathcal{O}$. Then, the results of example (2) above, conveniently adapted for a right Poisson action, confirm that $J_0 = R_{g_0} \circ J\colon x_0 \cdot G^* \to G$ is an equivariant momentum map for the transitive Poisson action $\bar{\rho}$. Consequently:

**5.2 Proposition.** *Let $(G, \pi_G)$ be a Poisson-Lie group, where $\pi_G$ is exact with linearization at the identity equal to $r \in \Lambda^2\mathfrak{g}$. Assume further that $r$ is invariant under the adjoint action of a closed Lie subgroup $H$ and that the center of $\mathfrak{h} = Lie(H)$ is equal to $\mathbf{R}X_0$, where $X_0 \in \mathfrak{h}$ is eventually zero. Then, all the simply connected symplectic leaves of the quotient Poisson space $G/H$ are universal covering spaces of right dressing orbits of $G^*$ in $G$.*

In particular, when $G = SU(2)$ viewed as a Poisson-Lie group as in [**15**] and $H = \mathbf{S}^1$ with the zero Poisson structure, the Poisson manifold $\mathcal{O}$ coincides with the 2-sphere $\mathbf{S}^2$. In this case, we have two symplectic leaves, a point and its complement (isomorphic to the plane), and the assumptions of Proposition 5.2 are fulfilled. Consequently, the open leaf is the universal covering space of a right dressing orbit of the three dimensional "book" group (the dual group of $SU(2)$) in $SU(2)$. More generally, if $G$ is a compact semisimple Lie group and $\mathcal{O}$ a coadjoint orbit of $G$, then according to [**15**], $G$ can be equipped with a Poisson-Lie structure which descends to $\mathcal{O}$ as a Poisson structure whose symplectic leaves are all cells of a Bruhat decomposition of $\mathcal{O}$ and diffeomorphic to dressing orbits of $G^*$ in $G$. The Propositions 2.2 and 5.2 provide a Hamiltonian interpretation of this situation and indicate a partial generalization for arbitrary Poisson-Lie group $G$.

**(4)** We discuss now an example where the conditions of Theorem 4.1 are always satisfied. Let us consider a semi-direct product $G = K \times_\rho \mathfrak{v}$, formed by a Lie group $K$ and a vector space $\mathfrak{v}$ through the representation $\rho\colon K \to GL(\mathfrak{v})$. The group law on $G$ is given by

$$(\kappa, u) \cdot (\lambda, v) = (\kappa\lambda, \kappa \cdot v + u), \tag{5.1}$$

for all $(\kappa, u), (\lambda, v) \in G$, where $\kappa \cdot v$ means the action of the element $\kappa \in K$ on $v \in \mathfrak{v}$ through the representation $\rho$ [**3**]. If $\phi\colon \mathfrak{v} \to \Lambda^2 T\mathfrak{v}$ is a Poisson-Lie structure on $\mathfrak{v}$ (for the abelian group law of a vector space) invariant under the representation $\rho$,

then the Poisson tensor $\pi = 0 \oplus \phi$ is a Poisson-Lie structure on $G$ for the group operation given by (5.1). Now, $\mathfrak{v}^*$ inherits a Lie algebra structure and we will denote by $V^*$ the corresponding connected and simply connected Lie group with Lie algebra $\mathfrak{v}^*$. The Lie group $V^*$ can obviously be seen as the dual group of the Poisson-Lie group $V = (\mathfrak{v}, \phi)$. If $\mathfrak{k}$ is the Lie algebra of $K$, then the dual group of $G$ is

$$G^* = \mathfrak{k}^* \times V^*$$

equipped with the direct product group operation. For $X \in \mathfrak{k}$ and $\kappa \in K$, one has two mappings:

$$\mathfrak{v}^* \to \mathfrak{v}^*, \quad q \mapsto X \cdot q, \forall q \in \mathfrak{v}^* \tag{5.2}$$

and

$$\mathfrak{v}^* \to \mathfrak{v}^*, \quad q \mapsto \kappa \cdot q, \forall q \in \mathfrak{v}^*. \tag{5.3}$$

In the previous equations, $X \cdot q$ and $\kappa \cdot q$ mean the actions of $\mathfrak{k}$ and $K$ respectively through the contragredient representation on $\mathfrak{v}^*$. The map (5.3) can be integrated to an action (by group homomorphisms) of the group $K$ on the dual group $V^*$. The induced fundamental vector fields coincide with the multiplicative vector fields obtained from (5.2) (observe that $q \mapsto X \cdot q$ is a 1-cocycle for the adjoint representation of $\mathfrak{v}^*$ on itself). If $\varrho\colon K \times V^* \to V^*$ denotes this action, then one obtains a linear map $\tau_p\colon \mathfrak{k} \to \mathfrak{v}^*$, for each $p \in V^*$, as follows:

$$\tau_p(X) = R_{p^{-1}} \varrho(X)_p. \tag{5.4}$$

We introduce the notation $\tau_p^*(v) = p \odot v$ for the transposed map $\tau_p^*\colon \mathfrak{v} \to \mathfrak{k}^*$. With the above data one can calculate the left dressing tranformation of $G$ on $G^*$ and the result is

$$\lambda_{(\kappa,a)}(\xi, p) = (\mathrm{Coad}(\kappa)\xi + \varrho_\kappa(p) \odot a, \varrho_\kappa(p)), \quad \forall (\kappa, a) \in G, (\xi, p) \in G^*. \tag{5.5}$$

For a given element $u_0 = (\xi_0, p_0) \in G^*$, it is easy to make the following observations using equation (5.5):

- If $K_{p_0}$ is the isotropy subgroup of $p_0$ with respect to the action $\varrho$ of $K$ on $V^*$, $\mathfrak{k}_{p_0}$ its Lie alebra and $i_{p_0}\colon \mathfrak{k}_{p_0} \hookrightarrow \mathfrak{k}$ the natural inclusion, then

$$\lambda_{(\kappa,a)}(i_{p_0}^* \xi, p_0) = (\mathrm{Coad}(\kappa) i_{p_0}^* \xi, p_0), \quad \forall (\kappa, a) \in K_{p_0} \times_\rho \mathfrak{v}.$$

- If $H = K_{p_0} \times_\rho \mathfrak{v}$, then $H^\circ = (\mathfrak{k}_{p_0})^\circ \times \{e\}$ and
$$u_0 \cdot H^\circ = (\xi_0 + (\mathfrak{k}_{p_0})^\circ, p_0).$$
- If $P$ is the orbit of $v_0 = i^* u_0 = (i^*_{p_0}\xi, p_0) \in H^*$ under the left dressing transformations of $H$, and $H_{v_0}$ the isotropy subgroup of $v_0$ with respect to the left dressing transformations of $H$ on $H^*$ , then
$$u_0 \cdot H^\circ = H_{v_0} \cdot u_0.$$

Now, in the case where $P$ is a dressing orbit in Theorem 4.1, we observe that $P_{ind}$ is a dressing orbit too (see also [**2**]). Combining this remark with the previous observations, we conclude that the dressing orbits of $G = K \times_\rho \mathfrak{v}$ (equipped with the Poisson-Lie structure $\pi = 0 \oplus \phi$) are obtained by Poisson induction on *coadjoint* orbits of certain subgroups of $K$.

Of particular interest is the case where $\mathfrak{v} = \mathfrak{k}^*$ and $\rho = \mathrm{Coad}$. Then $G = T^*K$ and $\pi$ coincides with the canonical Poisson structure on $T^*K$. Consequently, the dressing orbits of $T^*K$ can be obtained by Poisson induction on coadjoint orbits of subgroups of $K$.

**Acknowledgments.** I would like to thank J.-H. Lu for careful reading of the manuscript and critical and fruitful remarks. It is also a pleasure to thank M. Cahen and C. Duval for their kind interest in this work and useful suggestions.

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